Dynamic Performance Characteristics

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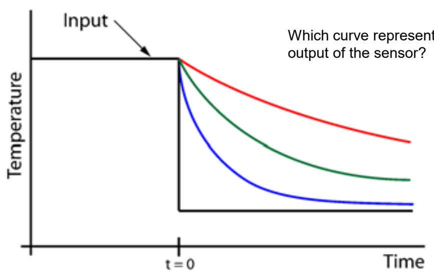
Dynamic Performance Characteristics

- Recall:
  - Static performance characteristics define sensor performance when the input and output are steady. We described steady state errors.
- Realistically, the sensor input changes all the time
- We must characterize the changing output of a sensor in response to a changing input to determine:
  - How fast the instrument responds (an important selection criterion)
  - Define the dynamic errors

Rapid Change in Input

- Consider an instantaneous change in the sensor input. What will happen with the output?



Mechanical System – Cup Anemometer

- Suppose the wind starts to blow. There is a finite amount of time required for the anemometer to come up to speed
- Similarly, the spinning cups carry kinetic energy which the anemometer must dissipate into the wind stream.
- Applying Newton's second law, we can write:

Thermal System – Thermometer

- We can describe heating or cooling using

- Where
  - c is the specific heat capacity of the sensor (J kg^{-1} K^{-1})
  - m is the mass of the sensor (kg)

Thermal System – Thermometer

- A thermometer is heated to a new temperature by conduction, so…

- Where
  - U is the heat transfer coefficient (J K^{-1} s^{-1} m^{-2})
  - A is the area of the sensor (m^2)
  - T is the temperature measured by the sensor (K)
  - T_{air} is the actual air temperature

Thermal System – Thermometer

Thermal System – Thermometer

- The constant has units of time, so we call it the time constant, τ

General form for a linear, first-order ordinary differential equation

$$\tau \frac{dx}{dt} + x = x_i$$

- τ = time constant
- Let x be some meteorological value
  - x = output value (this is what we want to solve for)
  - x_i = input value

Assumptions

- When determining dynamic performance characteristics, we make some assumptions:
  - Assume that the system has been calibrated to remove any bias
  - Assume that there are no static errors
  - Assume that if x_i = constant, then as $t \to \infty$, $x \to x_i$

$$\tau \frac{dx}{dt} + x = x_i$$

Time constant

- What exactly does the time constant tell us?

$$\frac{dx}{dt} = \frac{x_i - x}{\tau}$$

  - As τ increases, the sensor takes a longer time to respond
  - A smaller time constant indicates a sensor that responds faster to rapidly changing inputs

A special case of the equation for a thermometer

$$\tau \frac{dT}{dt} = T_{air} - T$$

- If the sensor input is the same as the sensor output (i.e., the static case), then T_{air} = T, so

$$\frac{dT}{dt} = 0$$

  - No temperature difference
  - No energy flow
  - No change in temperature

This is a static solution

Dynamic Solution

$$\tau \frac{dT}{dt} = T_{air} - T$$

What happens if $T_{air} \neq T$?

- In this case, $\frac{dT}{dt} \neq 0 \text{ for } \tau \neq 0$
- There are two parts to the solution of this differential equation…

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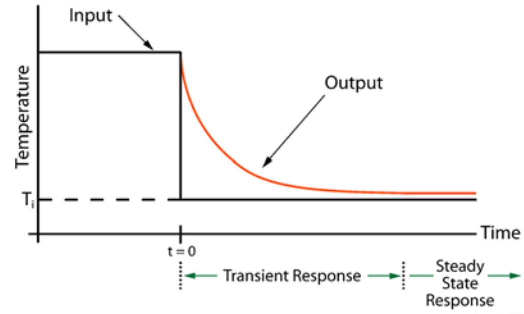
Dynamic Solution – Two Parts



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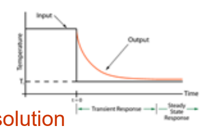
Dynamic Solution – Two Parts



- Transient solution
  - T(t) ≠ constant
  - Also called the homogeneous solution
- Steady-state solution
  - T(t) = constant
  - Also called the particular solution

$$x(t) = x_T(t) + x_S(t)$$

Transient Solution — Steady State Solution

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Dynamic Solution Procedure

1) First solve for the transient (or homogeneous) solution. This satisfies this equation:

$$\tau \frac{dx}{dt} + x = 0$$

2) Solve for the steady-state (particular) solution
3) Apply any initial conditions to solve for the coefficient

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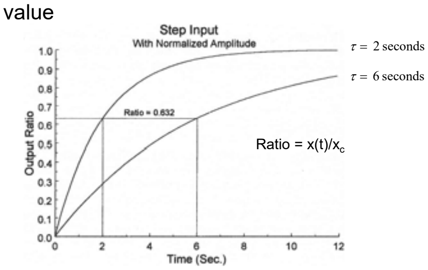
Let's do an example…

Wrapping up the example…

- The system requires roughly 5τ for us to consider the output as steady state!
- The larger the time constant, the longer it takes for the system to reach 63.2% of the final value
- The larger the time constant, the slower the instrument

Time constant

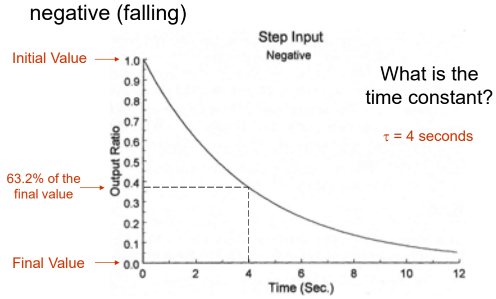
- Given a step change in input, the time constant (τ) is the time it takes for the output to reach 63.2% of the final value



Example: Hold a mercury thermometer over a flame. It's cool, but don't try it if you're older than 14.

Falling step function

- A step function can be either positive (rising) or negative (falling)



What is the time constant?

τ = 4 seconds

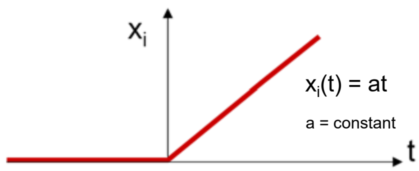
Example: Quickly move a thermometer from room temperature into an ice bath

Step Functions

- Ideal step functions don't exist in reality!
- All meteorological changes require a finite amount of time to reach a final value
- Changes can be fast (but how do we define "fast"?)
- A step function is a useful *approximation* if:

Time for step change << Time constant

Linear Ramp Function



- A linear ramp function represents an input that increases or decreases linearly over time

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Linear Ramp Function

- Recall the equation for a first-order, linear system:

$$\tau \frac{dx}{dt} + x = x_i$$

- For a ramp function, the input x_i is *at*, where *a* is a constant
- The equation becomes:

$$\tau \frac{dx}{dt} + x = at$$

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Linear Ramp Function

- Solve for the transient (homogeneous) solution by setting the right side equal to zero:

$$\tau \frac{dx}{dt} + x = 0$$

- We did this before!
- The transient solution is:

$$x_T(t) = C_1 e^{-t/\tau}$$

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Now for the hand-waving part…

- Solve for the particular solution by:
  - Assuming a linear solution (well, the input function is linear, so it's worth a shot…)
  - Test the assumed solution by plugging it back into the original governing equation
- Let's try this:

$$x_S(t) = k_0 + k_1 t$$

  - Now solve for both constants

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Linear Ramp Function
Steady-state solution

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D7

Linear Ramp Function
General solution

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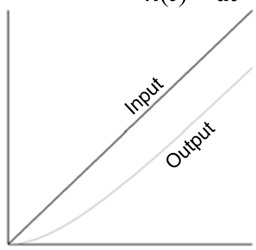
Linear Ramp Function
Example

- Consider the case where $x_0 = 0$. Then,

$$x(t) = at - a\tau\left(1 - e^{-t/\tau}\right)$$



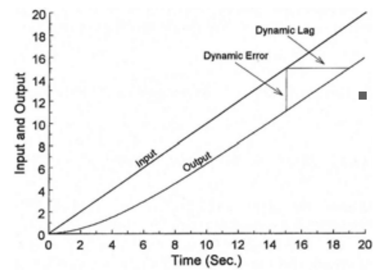
As t gets very large,

$$x_i(t) = at$$

$$x(t) = a(t - \tau)$$

Linear Ramp Function
Example

- The output never catches up with the input!



- Two errors appear:
  - Dynamic error
  - Dynamic lag

Dynamic Error

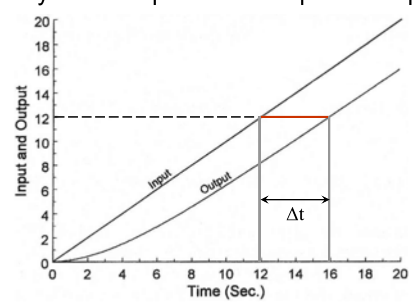
- Dynamic error is the difference between the ramp input and the response output at time *t*

$$\varepsilon_d = \underbrace{x(t)}_{\text{sensor output}} - \underbrace{x_i(t)}_{\text{sensor input}} = -a\tau\left(1 - e^{-t/\tau}\right)$$

- Let's consider the dynamic error both initially and at steady state
  - Initial dynamic error
    - For $t \to 0$: $\varepsilon_d = 0$
  - Steady-state dynamic error
    - For $t \to \infty$: $\varepsilon_d = -a\tau$

Dynamic Lag

- Dynamic lag is the time required for the steady-state output to catch up to the input



Dynamic Lag

- Consider the steady-state dynamic lag for a ramp function:

Sinusoidal Function

- In meteorology, we rarely see either step or ramp function inputs
- As you saw (or will see) in dynamics (ATMS 310), the motion of the atmosphere often resembles waves
- A simple wave is a sine wave
- Let's consider a sinusoidal input and solve for the steady-state output (we already found the transient response)

Sinusoidal Function

- Describe the sinusoidal input by:

$$x_i(t) = A_i \sin(\omega t)$$

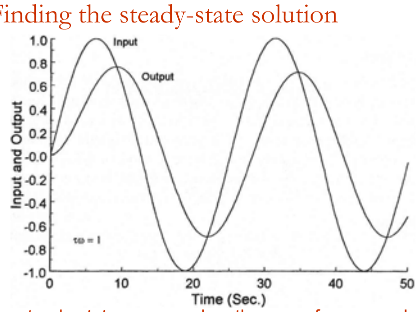
  - A_i is the amplitude (assumed constant)
  - ω is the oscillation frequency ($\omega = 2\pi f = 2\pi/T$); also constant
- Uh-oh…we have to assume a solution again

Sinusoidal Function

Finding the steady-state solution

Sinusoidal Function

Finding the steady-state solution



The steady-state response has the same frequency, but a different amplitude and phase, compared with the input.

Amplitude Ratio

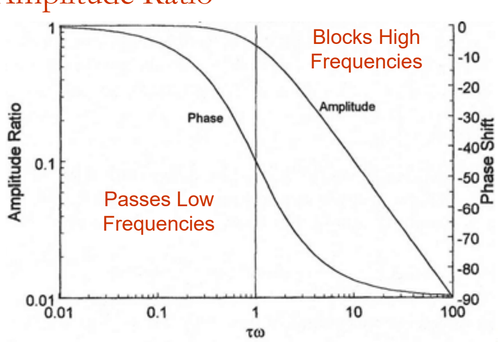
- Look at the ratio of the output amplitude to the input amplitude:

$$\frac{A}{A_i} = \frac{1}{\sqrt{1+(\tau\omega)^2}}$$

- A graph of this ratio shows that:
  - For low frequencies, the output amplitude is about the same as the input. Hence, these frequencies are passed through the system.
  - For high frequencies, the output amplitude is small compared with the input. Hence, these frequencies are *not* passed through the system

This is called a LOW-PASS FILTER

Amplitude Ratio



Amplitude Ratio

- For low frequencies, we pass nearly all amplitudes with little attenuation. This means that the sensor responds quickly to a slowly varying input. There is very little dynamic error.
- For high frequencies, we attenuate nearly all amplitudes. This means that the sensor responds slowly or not at all to a rapidly varying input. There is a large dynamic error.
- For a fast sensor:
  - $\tau << T$ $\tau\omega << 1$
  - almost no errors
- For a slow sensor:
  - $\tau >> T$ $\tau\omega >> 1$
  - large errors – perhaps no sensor output at all!

Superposition of Sinusoids

- What happens to the steady-state solution if there is more than one input frequency?

$$x_i(t) = A_0 + A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t)$$

- Superposition principle:
  - For a linear system, if $x_A(t)$ is a solution when $x_{iA}(t)$ is the input, and $x_B(t)$ is a solution when $x_{iB}(t)$ is the input, then $x_A(t)+x_B(t)$ is a solution when $x_{iA}(t)+x_{iB}(t)$ is the input

Superposition of Sinusoids

- For an input:

$$x_i(t) = A_0 + A_{i1} \sin(\omega_1 t) + A_{i2} \sin(\omega_2 t)$$

- The solution is