Formula Sheet 1 for Exam #2 **ATMS 310**

$$\frac{\mathbf{F}}{m} = -\frac{1}{\rho}\nabla p \qquad (1.1)$$

$$\frac{\mathbf{F}_g}{m} \equiv \mathbf{g}^* = -\frac{GM}{r^2}\left(\frac{\mathbf{r}}{r}\right) \qquad (1.3)$$

$$\mathbf{g}^* = \frac{\mathbf{g}_0^*}{(1+z/a)^2} \qquad (1.4)$$

$$\begin{aligned} F_{rx} &= \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \\ F_{ry} &= \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \qquad (1.5) \\ F_{rz} &= \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \end{aligned}$$

$$\frac{D\mathbf{V}}{Dt} = -\omega^2 r \qquad (1.6)$$

$$\mathbf{g} \equiv -g\mathbf{k} \equiv \mathbf{g}^* + \Omega^2\mathbf{R} \qquad (1.7)$$

$$\Phi = \int_0^z g\,dz \qquad (1.8)$$

$$\left(\frac{Du}{Dt}\right) = \left(2\Omega\sin\phi + \frac{u}{a}\tan\phi\right)\frac{Dy}{Dt} = 2\Omega v\sin\phi + \frac{uv}{a}\tan\phi \qquad (1.10a)$$

$$\left(\frac{Du}{Dt}\right) = -\left(2\Omega\cos\phi + \frac{u}{a}\right)\frac{Dz}{Dt} = -2\Omega w\cos\phi - \frac{uw}{a} \qquad (1.10b)$$

$$\left(\frac{Dv}{Dt}\right) = -2\Omega u \sin\phi - \frac{u^2}{a}\tan\phi \tag{1.11a}$$

$$\left(\frac{Dw}{Dt}\right) = 2\Omega u \cos\phi + \frac{u^2}{a} \tag{1.11b}$$

$$\left(\frac{Du}{Dt}\right)_{Co} = 2\Omega v \sin\phi = fv \tag{1.12a}$$

$$\left(\frac{Dv}{Dt}\right)_{Co} = -2\Omega u \sin\phi = -fu \tag{1.12b}$$

$$v = -fu_0 t \tag{1.13}$$

$$\left(\frac{D\mathbf{V}}{Dt}\right)_{Co} = -f\mathbf{k}\times\mathbf{V} \tag{1.14}$$

$$dp/dz = -\rho g \tag{1.18}$$

$$\Phi(z_2) - \Phi(z_1) = g_0(Z_2 - Z_1) = R\int_{p_2}^{p_1} T\,d\ln p \tag{1.21}$$

$$Z_T \equiv Z_2 - Z_1 = \frac{R}{g_0}\int_{p_2}^{p_1} T\,d\ln p \tag{1.22}$$

$$Z_T = H\ln(p_1/p_2) \tag{1.23}$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p = -\left(\frac{\partial \Phi}{\partial x}\right)_p \tag{1.25}$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p \tag{1.26}$$

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z \tag{1.27}$$

$$\left(\frac{\partial p}{\partial x}\right)_s = \left(\frac{\partial p}{\partial x}\right)_z + \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s\left(\frac{\partial p}{\partial s}\right) \tag{1.28}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) \tag{2.1}$$

$$\frac{D_a \mathbf{A}}{Dt} = \frac{D\mathbf{A}}{Dt} + \boldsymbol{\Omega} \times \mathbf{A} \tag{2.2}$$

$$\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F} \tag{2.3}$$

$$\begin{aligned} \frac{D_a \mathbf{U}_a}{Dt} &= \frac{D}{Dt}(\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}) \\ &= \frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R} \end{aligned} \tag{2.7}$$

$$\frac{D\mathbf{U}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{U} - \frac{\mathbf{1}}{\rho}\boldsymbol{\nabla} p + \mathbf{g} + \mathbf{F}_r \tag{2.8}$$

$$u \equiv r\cos\phi\,\frac{D\lambda}{Dt}, \quad v \equiv r\frac{D\phi}{Dt}, \quad w \equiv \frac{Dz}{Dt} \tag{2.9}$$

$$\frac{D\mathbf{U}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt} \tag{2.10}$$

$$\frac{D\mathbf{i}}{Dt} = \frac{u}{a\cos\phi}(\mathbf{j}\,\sin\phi - \mathbf{k}\cos\phi) \tag{2.11}$$

$$\frac{D\mathbf{j}}{Dt} = -\frac{u\tan\phi}{a}\,\mathbf{i} - \frac{v}{a}\,\mathbf{k} \tag{2.12}$$

$$\frac{D\mathbf{k}}{Dt} = \mathbf{i}\frac{u}{a} + \mathbf{j}\frac{v}{\mathbf{a}} \tag{2.13}$$

$$\begin{aligned}\frac{D\mathbf{U}}{Dt} &= \left(\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}\right)\mathbf{i} + \left(\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}\right)\mathbf{j} \\ &\quad + \left(\frac{Dw}{Dt} - \frac{u^2+v^2}{a}\right)\mathbf{k}\end{aligned} \tag{2.14}$$

$$\begin{aligned}-2\boldsymbol{\Omega}\times\mathbf{U} &= -2\Omega\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos\phi & \sin\phi \\ u & v & w\end{vmatrix} \\ &= -(2\Omega w\cos\phi - 2\Omega v\sin\phi)\,\mathbf{i} - 2\Omega u\sin\phi\,\mathbf{j} + 2\Omega u\cos\phi\,\mathbf{k}\end{aligned} \tag{2.15}$$

$$\nabla p = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z} \tag{2.16}$$

$$\mathbf{g} = -g\,\mathbf{k} \tag{2.17}$$

$$\mathbf{F}_r = \mathbf{i}F_{rx} + \mathbf{j}F_{ry} + \mathbf{k}F_{rz} \tag{2.18}$$

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx} \tag{2.19}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry} \tag{2.20}$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz} \tag{2.21}$$

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}; \qquad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.22}$$

$$\mathbf{V}_g \equiv \mathbf{k} \times \frac{1}{\rho f}\boldsymbol{\nabla} p \tag{2.23}$$

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right) \tag{2.24}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right) \tag{2.25}$$

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial z} - g &= -\frac{1}{(\rho_0 + \rho')}\frac{\partial}{\partial z}(p_0 + p') - g \\ &\approx \frac{1}{\rho_0}\left[\frac{\rho'}{\rho_0}\frac{dp_0}{dz} - \frac{\partial p'}{\partial z}\right] = -\frac{1}{\rho_0}\left[\rho' g + \frac{\partial p'}{\partial z}\right] \end{aligned} \tag{2.28}$$

$$\frac{\partial p'}{\partial z} + \rho' g = 0 \tag{2.29}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{U}) = 0 \tag{2.30}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \boldsymbol{\nabla} \cdot \mathbf{U} = 0 \tag{2.31}$$

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{1}{\rho \delta V}\frac{D}{Dt}(\rho \delta V) = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta V}\frac{D}{Dt}(\delta V) = 0 \tag{2.32}$$

$$\underset{\text{A}}{\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla}\rho'\right)} + \underset{\text{B}}{\frac{w}{\rho_0}\frac{d\rho_0}{dz}} + \underset{\text{C}}{\boldsymbol{\nabla} \cdot \mathbf{U}} \approx 0 \tag{2.33}$$

$$\boldsymbol{\nabla} \cdot (\rho_0 \mathbf{U}) = 0 \tag{2.34}$$

$$\frac{D}{Dt}\left[\rho\left(e + \frac{1}{2}\mathbf{U} \cdot \mathbf{U}\right)\delta V\right] = -\boldsymbol{\nabla} \cdot (p\mathbf{U})\delta V + \rho \mathbf{g} \cdot \mathbf{U}\delta V + \rho J \delta V \tag{2.35}$$

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{U} \bullet \mathbf{U} + \Phi\right) = -\mathbf{U} \bullet \boldsymbol{\nabla} p \tag{2.40}$$

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J \tag{2.41}$$

$$c_p \frac{D \ln T}{Dt} - R\frac{D \ln p}{Dt} = \frac{J}{T} \equiv \frac{Ds}{Dt} \tag{2.43}$$

$$\theta = T\,(p_s/p)^{R/c_p} \tag{2.44}$$

$$\frac{T}{\theta}\frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} \tag{2.47}$$

$$-\frac{dT}{dz} = \frac{g}{c_p} \equiv \Gamma_d \tag{2.48}$$

$$\frac{T}{\theta}\frac{\partial \theta}{\partial z} = \Gamma_d - \Gamma \tag{2.49}$$

$$\frac{Dw}{Dt} = \frac{D^2}{Dt^2}(\delta z) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{2.50}$$

$$\frac{D^2}{Dt^2}(\delta z) = -N^2 \delta z \tag{2.52}$$

where

$$N^2 = g\frac{d \ln \theta_0}{dz}$$

$$\frac{1}{\theta_0}\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) + w\frac{d \ln \theta_0}{dz} = \frac{J}{c_p T} \tag{2.53}$$

$$\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) + w\frac{d\theta_0}{dz} \approx 0 \tag{2.54}$$

$$\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+w\left(\Gamma_d-\Gamma\right)\approx 0 \tag{2.55}$$

$$\frac{D\mathbf{V}}{Dt}+f\mathbf{k}\times\mathbf{V}=-\boldsymbol{\nabla}_p\Phi \tag{3.2}$$

$$\begin{aligned}\frac{D}{Dt}&\equiv\frac{\partial}{\partial t}+\frac{Dx}{Dt}\frac{\partial}{\partial x}+\frac{Dy}{Dt}\frac{\partial}{\partial y}+\frac{Dp}{Dt}\frac{\partial}{\partial p}\\&=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial p}\end{aligned} \tag{3.3}$$

$$f\mathbf{V}_g=\mathbf{k}\times\boldsymbol{\nabla}_p\Phi \tag{3.4}$$

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p+\frac{\partial\omega}{\partial p}=0 \tag{3.5}$$

$$\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-S_p\omega=\frac{J}{c_p} \tag{3.6}$$

$$S_p\equiv\frac{RT}{c_pp}-\frac{\partial T}{\partial p}=-\frac{T}{\theta}\frac{\partial\theta}{\partial p} \tag{3.7}$$

$$S_p=\left(\Gamma_d-\Gamma\right)/\rho g$$

$$\frac{D\mathbf{V}}{Dt}=\mathbf{t}\frac{DV}{Dt}+\mathbf{n}\frac{V^2}{R} \tag{3.8}$$

$$\frac{D\mathbf{V}}{Dt}+f\mathbf{k}\times\mathbf{V}=-\frac{1}{\rho}\boldsymbol{\nabla}_p \tag{3.1}$$

$$\frac{DV}{Dt}=-\frac{\partial\Phi}{\partial s} \tag{3.9}$$

$$\frac{V^2}{R}+fV=-\frac{\partial\Phi}{\partial n} \tag{3.10}$$

$$fV_g = -\partial\Phi/\partial n \tag{3.11}$$

$$V^2/R + fV = 0 \tag{3.12}$$

$$V = \left(-R\frac{\partial\Phi}{\partial n}\right)^{1/2} \tag{3.14}$$

$$\begin{aligned} V &= -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} \\ &= -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{1/2} \end{aligned} \tag{3.15}$$

$$\frac{V_g}{V} = 1 + \frac{V}{fR} \tag{3.17}$$

$$\frac{Ds}{Dt} = V(x, y, t) \tag{3.18}$$

$$\frac{dy}{dx} = \frac{v(x, y, t_0)}{u(x, y, t_0)} \tag{3.19}$$

$$\frac{\partial\beta}{\partial t} = V\left(\frac{1}{R_t} - \frac{1}{R_s}\right) \tag{3.23}$$

$$R_t = R_s\left(1 - \frac{C\cos\gamma}{V}\right)^{-1} \tag{3.24}$$

$$\delta z \approx -g^{-1}RT\delta\ln p \tag{3.25}$$

$$p\frac{\partial v_g}{\partial p} \equiv \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p \tag{3.28}$$

$$p\frac{\partial u_g}{\partial p} \equiv \frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \tag{3.29}$$

or in vectorial form

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \boldsymbol{\nabla}_p T \tag{3.30}$$

$$\mathbf{V}_T \equiv \mathbf{V}_g(p_1) - \mathbf{V}_g(p_0) = -\frac{R}{f}\int_{p_0}^{p_1}\left(\mathbf{k} \times \boldsymbol{\nabla}_p T\right) d\ln p \tag{3.31}$$

$$u_T = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0); \quad v_T = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \tag{3.33}$$

$$\mathbf{V}_T = \frac{1}{f}\mathbf{k} \times \boldsymbol{\nabla}(\Phi_1 - \Phi_0) = \frac{g}{f}\mathbf{k} \times \boldsymbol{\nabla} Z_T = \frac{R}{f}\mathbf{k} \times \boldsymbol{\nabla}\langle T\rangle \ln\left(\frac{p_0}{p_1}\right) \tag{3.35}$$

$$\omega = \frac{\partial p}{\partial t} + \mathbf{V}_a \bullet \boldsymbol{\nabla} p - g\rho w \tag{3.37}$$

$$\omega = -\rho g w \tag{3.38}$$

$$w(z) = \frac{\rho(z_s)\, w(z_s)}{\rho(z)} - \frac{p_s - p}{\rho(z) g}\left(\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y}\right) \tag{3.40}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \frac{u(x_0 + d) - u(x_0 - d)}{2d} + \frac{v(y_0 + d) - v(y_0 - d)}{2d} \tag{3.41}$$

$$\omega = S_p^{-1}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) \tag{3.42}$$

$$[q(x, t + \delta t) - q(x, t - \delta t)]/(2\delta t) = -c\,[q(x + \delta x, t) - q(x - \delta x, t)]/(2\delta x) \tag{13.7}$$

$$\sigma = c\delta t/\delta x \leq 1 \tag{13.18}$$

$$\frac{\partial p_s}{\partial t} \approx -\int_0^{p_s} (\nabla \bullet \mathbf{V})\, dp \quad (3.44)$$

$$\frac{\partial \Phi_s}{\partial t} \approx -\frac{RT_s}{p_s} \int_0^{p_s} (\nabla \bullet \mathbf{V})\, dp \quad (3.45)$$

$$C \equiv \oint \mathbf{U} \bullet d\mathbf{l} = \oint |\mathbf{U}| \cos\alpha \, dl$$

$$\frac{DC_a}{Dt} = \frac{D}{Dt} \oint \mathbf{U}_a \bullet d\mathbf{l} = -\oint \rho^{-1} dp \quad (4.3)$$

$$C = C_a - C_e = C_a - 2\Omega A_e \quad (4.4)$$

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt} \quad (4.5)$$

$$C_2 - C_1 = -2\Omega\,(A_2 \sin\phi_2 - A_1 \sin\phi_1) \quad (4.6)$$

$$\frac{D\langle v \rangle}{Dt} = \frac{R \ln(p_0/p_1)}{2(h+L)} (\overline{T}_2 - \overline{T}_1) \quad (4.7)$$

$$\eta \equiv \mathbf{k} \bullet (\nabla \times \mathbf{U}_a), \qquad \zeta \equiv \mathbf{k} \bullet (\nabla \times \mathbf{U})$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$$

$$\mathbf{k} \bullet \nabla \times \mathbf{U}_e = 2\Omega \sin\phi \equiv f$$

$$\zeta \equiv \lim_{A \to 0} \left(\oint \mathbf{V} \bullet d\mathbf{l} \right) A^{-1} \quad (4.8)$$

$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{(\delta n\, \delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s} \quad (4.9)$$

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{Const} \qquad (4.12)$$

$$(\zeta + f)/h = \eta/h = \text{Const} \qquad (4.13)$$

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \qquad (4.17)$$

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (4.22b)$$

$$\frac{\partial \zeta_\theta}{\partial t} = -\nabla_\theta \bullet \left[(\zeta_\theta + f)\mathbf{V} - \left(\mathbf{F}_r - \dot{\theta}\frac{\partial \mathbf{V}}{\partial \theta}\right) \times \mathbf{k}\right] \qquad (4.38)$$

$$\frac{D_g \mathbf{V}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{V}_a - \beta y \mathbf{k} \times \mathbf{V}_g \qquad (6.11)$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \qquad (6.12)$$

$$f_0 v_g = \frac{\partial \Phi}{\partial x}, \quad f_0 u_g = -\frac{\partial \Phi}{\partial y} \qquad (6.14)$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi \qquad (6.15)$$

$$\frac{D_g \zeta_g}{Dt} = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g \qquad (6.18)$$

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\zeta_g + f\right) + f_0 \frac{\partial \omega}{\partial p} \tag{6.19}$$

$$\frac{1}{f_0}\boldsymbol{\nabla}^2 \chi = -\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\frac{1}{f_0}\boldsymbol{\nabla}^2 \Phi + f\right) + f_0 \frac{\partial \omega}{\partial p} \tag{6.21}$$

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)\right] - f_0 \frac{\partial \omega}{\partial p} - f_0 \frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right) \tag{6.22}$$

$$\underbrace{\left[\boldsymbol{\nabla}^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]}_{\text{A}} \chi = -\underbrace{f_0 \mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\frac{1}{f_0}\boldsymbol{\nabla}^2 \Phi + f\right)}_{\text{B}} \\ \underbrace{-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{C}} \tag{6.23}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \bullet \boldsymbol{\nabla}\right) q = \frac{D_g q}{Dt} = 0 \tag{6.24}$$

where q is the quasi-geostrophic potential vorticity defined by

$$q \equiv \left[\frac{1}{f_0}\boldsymbol{\nabla}^2 \Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] \tag{6.25}$$

$$\underbrace{\left(\boldsymbol{\nabla}^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{\text{A}} \omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\underbrace{\left[\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\frac{1}{f_0}\boldsymbol{\nabla}^2 \Phi + f\right)\right]}_{\text{B}} \\ + \underbrace{\frac{1}{\sigma}\boldsymbol{\nabla}^2\left[\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{C}} - \underbrace{\frac{\kappa}{\sigma p}\boldsymbol{\nabla}^2 J}_{\text{D}} \tag{6.34}$$

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{\text{A}}\omega \approx \frac{f_0}{\sigma}\left[\frac{\partial \mathbf{V}_g}{\partial p} \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] \tag{6.36}$$

RHS (6.36) $\sim \dfrac{f_0}{\sigma}\left(-\boldsymbol{V_T} \cdot \nabla\left[\dfrac{\nabla^2\Phi}{f_0} + f \quad \right]\right)$

$$\sigma\,\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla \bullet \mathbf{Q} + f_0\beta\frac{\partial v_g}{\partial p} - \frac{\kappa}{p}\nabla^2 J \tag{6.53}$$

$$\mathbf{Q} = -\frac{R}{p}\left(\frac{\partial T}{\partial y}\right)\left(\frac{\partial v_g}{\partial x}\mathbf{i} - \frac{\partial u_g}{\partial x}\mathbf{j}\right)$$