The Vorticity Equation describes the factors that can alter the magnitude of the absolute vorticity with time.

### Vorticity Equation in Cartesian Coordinates

The \((x,y,z,t)\) form is derived from the primitive horizontal equations of motion:

\[
\begin{align*}
\frac{\partial v}{\partial t} &= -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} - \alpha \frac{\partial p}{\partial y} - fu + F_{xy} \quad (1) \\
\frac{\partial u}{\partial t} &= -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - w \frac{\partial u}{\partial z} - \alpha \frac{\partial p}{\partial x} + fv + F_{xz} \quad (2)
\end{align*}
\]

where:

- a) local rate of change of the meridional or zonal velocity components
- b) advection terms
- c) pressure gradient force
- d) Coriolis force
- e) Friction force

The derivation will not be carried out here. The method is as follows:

1) Take \(\frac{\partial}{\partial x}\) of Equation (1)
2) Take \(\frac{\partial}{\partial y}\) of Equation (2)
3) Subtract the result of step 2 from the result of step 1

After use of the product rule, simplifications, and cancellations, the result is the vorticity equation in \((x,y,z,t)\) coordinates:

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} &= -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - w \frac{\partial \zeta}{\partial z} - (\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \right) + \frac{\partial F_{xy}}{\partial y} + \frac{\partial F_{xz}}{\partial y} \\
&\quad + \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right) \quad (3)
\end{align*}
\]

The terms of the vorticity equation are defined as follows:
a) Rate of change of relative vorticity at a grid point
b) Zonal advection of relative vorticity
c) Meridional advection of relative vorticity

*Note that if terms b) and c) increase the vorticity, we call that “Positive Vorticity Advection” or PVA. If they work to decrease the vorticity, it is called “Negative Vorticity Advection” or NVA.*

d) Vertical transfer of relative vorticity
e) Advection of planetary vorticity

*Note that planetary vorticity is advected by the meridional component of the wind, since planetary vorticity only changes in the meridional direction (function of latitude)*

Air moving northward from the Equator will encounter increasing planetary vorticity, causing a curve to the right or more anti-cyclonic motion. This will decrease the local rate of change of vorticity since anti-cyclonic spin is negative vorticity. Southward moving air will encounter decreasing planetary vorticity, causing the air to move more to the left (cyclonically, increasing the rate of change of vorticity).

f) Divergence Term

If the horizontal wind divergence > 0, the change in vorticity will be downward. If there is horizontal wind convergence, that will impart more positive vorticity.

g) Tilting Terms

*Essentially this term represents the change in vertical velocity in the horizontal direction. If air is rising faster in one area than another, this imparts a tilting effect, which creates spin in the atmosphere*

h) Friction terms
i) Solenoidal Terms

*Represents the effects of the pressure gradient acceleration. If the pressure gradient varies in such a way to produce clockwise rotation, this means a negative change in the relative vorticity (and vice versa).*

Holton combines the partial terms in Eq. (3), does not consider the friction terms, and substitutes for the \( \nu(\partial f/\partial y) \) term to yield Eq. 4.17 on p. 101:

\[
\frac{D}{Dt} \left( \zeta + f \right) = - \left( \zeta + f \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right)
\]  

\[ (4) \]
Vorticity Equation in Isobaric Coordinates

The steps to derive the vorticity equation in isobaric coordinates are the same as Cartesian, except one begins with the isobaric equations of motion. The resulting equation is:

\[
\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - \omega \frac{\partial \zeta}{\partial \rho} - v \frac{\partial f}{\partial y} \left( \zeta + f \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left( \frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial \rho} \frac{\partial \rho}{\partial y} \right) + \left( \frac{\partial F_{xy}}{\partial x} - \frac{\partial F_{ry}}{\partial y} \right)
\]

All terms remain the same EXCEPT the solenoidal terms drop out. Recall that the solenoidal terms represented the effects of the pressure gradient force on the vorticity. For an isobaric surface, there are no pressure gradients. Thus, there cannot be a solenoidal term.

Application of the Vorticity Theorem

In 1988, the following paper was published in *Monthly Weather Review*, a professional publication put out by the American Meteorological Society:


They examined 20 extratropical cyclones – 10 developed normally, and 10 were categorized as “rapidly deepening” (aka “BOMBS”), defined as a surface pressure fall of at least 24 mb over 24 hours. They calculated the terms of the vorticity theorem to try to find the important contributors to the rapidly deepening storms.

Regular cyclones had:
1) Divergence term produced significant contributions to the vorticity only in the planetary boundary layer (high convergence -> more vorticity)
2) Vertical vorticity advection not important
3) PVA throughout most of the troposphere
4) Little change in vorticity at all levels as storm moves from incipient stage to the mature stage.

Bombs had:
1) High positive and negative values of the divergence term throughout the entire troposphere, with a sharp reversal of sign near the 500 mb level
2) Vertical vorticity advection positive (upward) throughout the troposphere
3) PVA large only in the upper troposphere
4) A dramatic increase in vorticity at all levels as storm moved from its incipient to mature phase
They conclude that bomb generation depends “critically on the generation of positive vorticity by the convergent flow in the lower troposphere. Such a process necessitates the preexistence of atmospheric volumes containing significant amounts of positive vorticity which can be drawn into the region of incipient cyclogenesis”.

**Scale Analysis of the Vorticity Equation**

We can examine the relative magnitudes of each term in the vorticity equation through a scale analysis.

First, it can be shown that the relative vorticity \( \zeta \) is often small compared to the absolute vorticity:

\[
\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \frac{U}{L} \equiv \frac{10^6 \text{ m}}{10^5 \text{ m}} \equiv 10^{-5} \text{ s}^{-1}
\]

\[
f \approx 10^{-4} \text{ s}^{-1}
\]

Thus, \( \zeta \) may be neglected in the divergence term of the vorticity equation:

\[
(\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \approx f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]

Note that near the center of intense cyclonic storms, \( \zeta \) approaches \( 10^{-4} \) and thus we cannot make this simplification since it is of similar magnitude to \( f \).

Similar scale analyses can be performed on the other terms of the vorticity equation using typical mid-latitude synoptic-scale values given on p. 104 of Holton:

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} & = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - w \frac{\partial \zeta}{\partial z} - v f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \right) \\
& \quad + \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial x} \right) \\
& \approx 10^{-10} 10^{-10} 10^{-10} 10^{-11} 10^{-10} 10^{-9} 10^{-9} 10^{-11}
\end{align*}
\]

All units of these terms are \( \text{s}^{-2} \). If we retain only the terms that are \( 10^{-10} \) are greater, we get a form of the vorticity equation valid for synoptic-scale motions:

\[
\frac{D}{Dt} (\zeta + f) = -f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad (5)
\]
For intense cyclonic storms, the relative vorticity must be added back in to the divergence term on the right hand side of Eq. (5).

What does Equation 5 say in words? Basically,

“The change in absolute vorticity following the horizontal motion on the synoptic scale is given by the concentration or dilution of planetary vorticity caused by the convergence or divergence of the horizontal flow.”

_Divergence_ along the flow will lower the absolute vorticity of the flow. _Convergence_ along the flow will increase the absolute vorticity of the flow.

Equation 5 also helps to explain why cyclones are much more intense than anti-cyclones.

- If there is convergence, this will increase $\zeta$, which will increase the relative vorticity, which leads to even more cyclonic vorticity.

- If there is divergence, this will decrease $\zeta$. Eventually, the relative vorticity will become negative and exactly opposite of the planetary vorticity ($f$). From then on, the absolute vorticity will not change, no matter how much more divergence occurs in the flow.