Conservation of Energy
Thermodynamic Energy Equation

The previous two sections dealt with conservation of momentum (equations of motion) and the conservation of mass (continuity equation). This section addresses the conservation of energy.

The first law of thermodynamics, of which you should be very familiar with from thermodynamics class, is the conservation of energy law. The most common form of the 1st law in meteorology is the enthalpy form, which is written as:

\[ dq = C_p \, dT - \alpha \, dp \]  

(1)

where \( dq \) = change in heating, \( C_p dT \) = change in internal energy (where \( C_p \) is the specific heat at constant pressure), and \( -\alpha \, dp \) = change in energy due to the work of expansion.

Adding \( \alpha \, dp \) to both sides of (1) and then dividing by \( C_p \, dt \) yields:

\[ \frac{dT}{dt} = \frac{\alpha}{C_p} \, \frac{dp}{dt} + \frac{1}{C_p} \, \frac{dq}{dt} \]  

(2)

with the specified terms defined as:

a) rate of change of temperature (T) with time inside an air parcel
b) rate of change of temperature with time due to work of expansion
c) rate of change of T with time due to diabatic heating

This is nice, but we probably want a more useful form if we are trying to predict temperature in a numerical model.

Now we know from Euler’s relation that the total derivative (a) can be broken down into the local rate of change and advection components. After the substitution and some rearranging, Eq. (2) becomes:

\[ \frac{\partial T}{\partial t} = -u \, \frac{\partial T}{\partial x} - v \, \frac{\partial T}{\partial y} - w \, \frac{\partial T}{\partial z} + \frac{\alpha}{C_p} \, \frac{dp}{dt} + \frac{1}{C_p} \, \frac{dq}{dt} \]  

(3)

a) rate of change of T at a grid point – Remember, this is the Eulerian form now
b) zonal advection of T – The sign of this term depends on the sign of the temperature gradient. Is warmer or cooler air being moved in?
c) meridional advection of T – Usually, temperature decreases as latitude increases \((\frac{\partial T}{\partial y} < 0)\). So if there is a south wind \((v > 0)\), this term will contribute positively to the local temperature change. Warm air is being advected in.
d) vertical transfer of $T$ – In most cases the temperature decreases with height in the atmosphere ($\frac{dT}{dz} < 0$). Upward velocity ($w > 0$) will advect warmer air.

e) adiabatic temperature change

f) diabatic temperature change

**Thermodynamic Energy Equation**

If you watch the Weather Channel, you will frequently hear the meteorologist say “There is a lot of energy associated with this system”, or “Most of the energy is concentrated to the north of the frontal boundary”. Are they referring to the winds? Probably not. Usually they are alluding to an area with intense precipitation and/or convection – an area with a lot of latent heat release.

Heating and kinetic energy are intimately linked. For example, if an air parcel is warmed, it becomes buoyant and begins to rise, acquiring kinetic energy. Also, heating affects the kinetic energy of molecules at a microscopic level. Molecules will vibrate more vigorously as they are heated. We have a term called thermodynamic energy which wraps up this internal energy (vibrating molecules) with the kinetic energy (wind). It is defined as the sum of these two components.

The equation which describes this relationship, called the thermodynamic energy equation, is derived beginning with an alternative form of the 1\textsuperscript{st} Law of Thermodynamics, the internal energy form:

$$dq = du_e + pd\alpha$$  \hspace{1cm} (4)

After some rearranging, dividing by $dt$, and substituting $\alpha = 1/\rho$ we get:

$$\frac{du_e}{dt} = -p \frac{d}{dt} \left[ \frac{1}{\rho} \right] + \frac{dq}{dt}$$ \hspace{1cm} (5)

Mathematics tells us that $\frac{d}{dt} \left[ \frac{1}{\rho} \right] = -\frac{1}{\rho^2} \frac{d\rho}{dt}$. Equation (5) becomes:

$$\frac{du_e}{dt} = \frac{p}{\rho^2} \frac{d\rho}{dt} + \frac{dq}{dt}$$ \hspace{1cm} (6)

Now multiply both sides by $\rho$: 
\[
\rho \frac{du_e}{dt} = \frac{p}{\rho} \frac{d\rho}{dt} + \rho \frac{dq}{dt} \quad (7)
\]

Now things get really silly. Do you see anything in the first term on the right side that can be substituted for? Hint? Well, here’s the continuity equation that you have already forgotten about:

\[
\frac{1}{\alpha} \frac{d\rho}{dt} = -\nabla \cdot \vec{V}
\]

or,

\[
\frac{d\rho}{dt} = \rho \frac{dq}{dt}
\]

Now do you see anything we can substitute for? Then do it! Eq. (7) becomes:

\[
\rho \frac{du_e}{dt} = -p\nabla \cdot \vec{V} + \rho \frac{dq}{dt} \quad (8)
\]

This equation is called the **thermal energy equation**, and it is worth a couple minutes to point out some things. It tells us how the internal energy changes with time (per unit volume). The change of internal energy depends upon the rate at which work is done (\(- p\nabla \cdot \vec{V}\) term) on the volume and the rate at which energy is added (\(\rho \frac{dq}{dt}\)).

The work done on the volume is related to the velocity divergence. If there is a net divergence (\(\nabla \cdot \vec{V} > 0\)), equation 8 tells us that the change in internal energy < 0. In other words, divergence leads to expansion of the volume (which requires work), lowering the internal energy. The opposite can be said of velocity convergence, which will warm the parcel.

The thermal energy equation makes up half of the thermodynamic energy equation. Now we need to find a relationship for the change in mechanical (kinetic energy) with heating. Where better to start than Newton’s second law of motion in vector form!

\[
\frac{d\vec{V}}{dt} = -\alpha \vec{V} - 2\Omega \times \vec{V} + \vec{g} + \vec{F}_r \quad (9)
\]

Multiply both sides of (9) by \(\rho\) and then dot multiply both sides by \(\vec{V}\) to yield:

\[
\vec{V} \cdot \left(\rho \frac{d\vec{V}}{dt}\right) = \vec{V} \cdot (-\nabla p) + \vec{V} \cdot (-\rho 2\Omega \times \vec{V}) + \vec{V} \cdot \vec{g} + \vec{V} \cdot \rho \vec{F}_r \quad (10)
\]
Now we can simplify a bit. The rotation term \( \vec{V} \cdot (- \rho 2\Omega \times \vec{V}) \) becomes = 0 because the cross product creates a third vector that is perpendicular to the original two vectors. Dot multiplying the wind vector with the perpendicular vector = 0. I know, you are uttering explicatives right now but go back to the math section if you need to review.

The term \( \vec{V} \cdot \rho \vec{g} \) can be written as:

\[
\left( u \vec{i} + v \vec{j} + w \vec{k} \right) \cdot \left( - \rho \vec{g} \right)
\]

since gravity only acts in the k direction (and is negative because it acts in the negative k direction). When you carry out the dot product, remember that a unit vector that is dotted with itself = 1 and one dotted with any other unit vector = 0. Thus, (11) becomes:

\[
\left( u \vec{i} + v \vec{j} + w \vec{k} \right) \cdot \left( - \rho \vec{g} \right) = - \rho g w
\]

Now things get really fun. We know that the vertical velocity (w) = dz/dt. Also, try to recall something else that you probably already forgot, which is that gdz = d\( \Phi \) (change in geopotential). So,

\[
- \rho g w = - \rho g \frac{dz}{dt} = - \rho \frac{d\Phi}{dt}
\]

This term is now essentially the “gravitational potential energy”. Energy is needed to lift parcels into the atmosphere, and this term accounts for that. Substituting (13) into (10), we get:

\[
\vec{V} \cdot \left( \rho \frac{d\vec{V}}{dt} \right) = \vec{V} \cdot (- \nabla \rho) - \rho \frac{d\Phi}{dt} + \vec{V} \cdot \rho \vec{F}_r
\]

We are almost home. The term on the LHS of (14) is the kinetic energy and can be rewritten as:

\[
\vec{V} \cdot \left( \rho \frac{d\vec{V}}{dt} \right) = \rho \frac{d\left( \frac{1}{2} \vec{V} \cdot \vec{V} \right)}{dt}
\]

Substitute (15) into (14) and rearrange a little to get:

\[
\rho \frac{d\left( \frac{1}{2} \vec{V} \cdot \vec{V} \right)}{dt} + \rho \frac{d\Phi}{dt} = \vec{V} \cdot (- \nabla \rho) + \vec{V} \cdot \left( \rho \vec{F}_r \right)
\]

Finally, combine the two terms on the LHS to yield:
This is the mechanical energy equation. In English, it says that the sum of the kinetic and geopotential energy change with time is equal to the production of energy by the PGF (b) and the dissipation of energy from the friction force (c). Note that the units for all terms in this equation are the same as the thermal energy equation.

Now we want to combine the thermal energy equation (8) with the mechanical energy equation (17) to create the thermodynamic energy equation. Recall the thermal energy equation:

\[
\rho \frac{du_c}{dt} = -p \nabla \cdot \vec{V} + \rho \frac{dq}{dt}
\]  

(8)

Add this to equation (17) to yield:

\[
\rho \frac{du_c}{dt} + \frac{d}{dt} \left( \frac{1}{2} \vec{V} \cdot \vec{V} + \Phi \right) = -p \nabla \cdot \vec{V} + \vec{V} \cdot (\nabla p) + \vec{V} \cdot \left( \rho \vec{F}_r \right) + \rho \frac{dq}{dt}
\]

(18)

You can entertain yourself by showing that:

\[-p \nabla \cdot \vec{V} + \vec{V} \cdot (\nabla p) = -\nabla \cdot (p \vec{V})\]

Substituting, rearranging, and moving the gravitational potential energy back to the RHS gives us:

\[
\rho \frac{du_c}{dt} + \frac{d}{dt} \left( \frac{1}{2} \vec{V} \cdot \vec{V} \right) = -\nabla \cdot (p \vec{V}) + \vec{V} \cdot \left( \rho \vec{F}_r \right) + \rho \frac{dq}{dt} - \rho \frac{d\Phi}{dt}
\]

(19)

Now we can combine the two terms on the LHS, multiply through by volume (\(\delta V\)), and revert the \(-\rho \frac{d\Phi}{dt}\) to \(\vec{V} \cdot \rho \vec{g}\) gives us a form of the thermodynamic energy equation:
a) rate of change of the internal energy and kinetic energy
b) gravitational acceleration
c) pressure gradient force
d) frictional deceleration
e) diabatic heating

So we see that internal and kinetic energy is directly controlled by the not only the forces of gravity, friction, and the PGF but also by the amount of heating that occurs within a given volume.