The Static Atmosphere
Equation of State, Hydrostatic Balance, and the Hypsometric Equation

Equation of State

The atmosphere at any point can be described by its temperature, pressure, and density. These variables are related to each other through the Equation of State for dry air:

\[ p\alpha = R_d T_v \]  

(1)

Where \( \alpha \) is the specific volume \((1/\rho)\), \( p \) = pressure, \( T_v \) = virtual temperature \((\text{K})\), and \( R \) is the gas constant for dry air \((R = 287 \text{ J kg}^{-1} \text{ K}^{-1})\). The virtual temperature is the temperature air would have if we account for the moisture content of it. Above the surface, where moisture is scarce, \( T_v \) and \( T \) are nearly equal. Near the surface you can use the following equation to calculate \( T_v \):

\[ T_v = T(1 + 0.6078q) \]

where \( q \) = specific humidity (\( T \) must be in K, \( q \) must be dimensionless). More on virtual temperature a little later.....

We can also write the equation of state as:

\[ p = \rho R_d T_v \]  

(2)

The equation of state is also known as the “Ideal Gas Law”. The atmosphere is treated as an “Ideal Gas”, meaning that the molecules are considered to be of negligible size so that they exert no intermolecular forces.

Hydrostatic Equation

Except in smaller-scale systems such as an intense squall line, the vertical pressure gradient in the atmosphere is balanced by the gravity force. We call this the hydrostatic balance.

\[ \frac{dp}{dz} = -\rho g \]  

(3)
Integrating (3) from a height $z$ to the top of the atmosphere yields:

$$p(z) = \int_z^{\infty} \rho g dz$$  \hspace{1cm} (4)

This says that the pressure at any point in the atmosphere is equal to the weight of the air above it.

**Hypsometric Equation**

In our discussion of effective gravity we mentioned geopotential ($\Phi$). It is the work that must be done to raise an object 1 km against the earth’s gravitational field. The geopotential at height $z$ is given by:

$$\Phi(z) = \int_0^z g dz$$  \hspace{1cm} (5)

At sea level ($z = 0$), $\Phi = 0$ by convention.

The geopotential height ($Z$) is found by dividing (5) by the gravitational acceleration at the surface of the earth ($g_o$):

$$Z \equiv \frac{\Phi(z)}{g_o} = \frac{1}{g_o} \int_0^z g dz$$  \hspace{1cm} (6)

For an expanded discussion of geopotential and geopotential height, see the pdf notes on the class website. If we take the derivative of both sides of (5) we get:

$$d\Phi = g dz$$  \hspace{1cm} (7)

Using the equation of state and the hydrostatic approximation it can be shown that:
\[ gdz = -\frac{dp}{p} R_d T_v \]  

Here \( R_d \) is the gas constant for dry air (no moisture), and \( T_v \) is called the virtual temperature. This is a temperature that is corrected from the real air temperature to account for changes in density of the air due to moisture. See the following for a discussion on how this is done:

http://www.cimms.ou.edu/~doswell/virtual/virtual.html

If we substitute (8) into (7) and then integrate between two pressure levels \( p_1 \) and \( p_2 \), we get:

\[ \Phi_2 - \Phi_1 = -R_d \int_{p_1}^{p_2} T_v \frac{dp}{p} \]  

(9)

We can use an average value of \( T_v \) over the layer \( p_1 \) to \( p_2 \). This allows us to pull it out of the integral. In addition, divide both sides by \( g_o \) to convert geopotential to geopotential height. (9) becomes:

\[ Z_2 - Z_1 = \frac{R_d T_v}{g_o} \int_{p_1}^{p_2} \frac{dp}{p} \]  

(10)

Reversing the limits of integration and then performing the integration we get:

\[ Z_2 - Z_1 = \frac{R_d T_v}{g_o} \ln \left( \frac{p_1}{p_2} \right) \]  

(11)

Equation (11) is one form of the hypsometric equation. The quantity \((Z_2 - Z_1)\) is called the “thickness” of the layer between pressure levels \( p_1 \) and \( p_2 \) and is usually represented by \( Z_T \). The hypsometric equation tells us that the thickness of a layer between two pressure levels is proportional to the mean temperature of that layer. For colder (warmer) layers, the pressure decreases more (less) rapidly with height.

The quantity \( \frac{R_d T_v}{g_o} \) is defined as the scale height (H). Also, the thickness is represented as \( Z_T \). Eq. (11) becomes:

\[ Z_T = H \ln \left( \frac{p_1}{p_2} \right) \]  

(12)

From (12) we can see that pressure decreases exponentially with geopotential height in an isothermal atmosphere by a factor of 1/e per scale height.
\[ p(Z) = p(0)e^{-\frac{Z}{H}} \] (13)