ATMS 310
Potential Vorticity

Potential Vorticity is defined as a variable that combines the absolute vorticity and some measure of the thickness of a column of air.

Derivation

Potential temperature is defined as:

$$\theta = T\left(\frac{p_s}{p}\right)^{\frac{R}{c_p}} \qquad (1)$$

This is called Poisson's equation. The variable p_s is the standard pressure at which we measure the air parcel's temperature if taken there dry adiabatically from pressure level p (usually p_s = 1000 hPa). Using the Equation of State, we can solve for the density ρ:

$$\rho = p^{\frac{c_v}{c_p}} (R\theta)^{-1} (p_s)^{\frac{R}{c_p}} \qquad (2)$$

Thus, on an isentropic surface (temperature is constant, or constant θ surface), density is only a function of pressure. In this case, the solenoidal term from the circulation theorem (Eq. 2 from Circulation/Vorticity notes) is zero:

$$\oint \frac{dp}{\rho} = 0 \qquad (3)$$

In this barotropic situation, the absolute circulation is conserved following the motion. This satisfies what is known as the Kelvin circulation theorem, which can be expressed as:

$$\frac{D}{Dt}\left(C + 2\Omega \delta A \sin\phi\right) = 0 \qquad (4)$$

where C = circulation evaluated for a closed loop encompassing the area δ A on an isentropic surface. The circulation (C) is approximated by the relative vorticity multiplied by the area encompassed, or:

$$C \approx \zeta \delta A \qquad (5)$$

Assuming an infinitesimal parcel of air and substituting (5) into (4),

$$\delta A\left(\zeta_\theta + f\right) = Const \qquad (6)$$

where ζ_θ is the vertical component of relative vorticity on an isentropic surface and f is the Coriolis parameter ($f = 2\Omega \sin\phi$).

Study the figure below (Fig. 4.7 in Holton) – it represents an air parcel confined between two potential temperature surfaces, separated by a pressure interval δp :

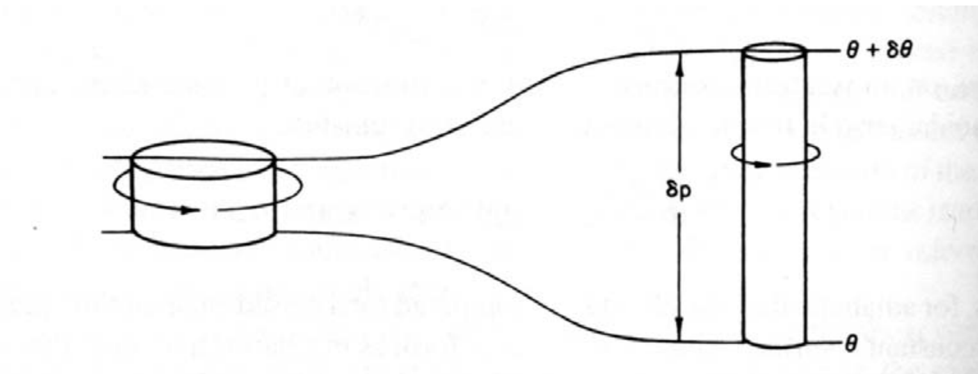


Fig. 4.7 A cylindrical column of air moving adiabatically, conserving potential vorticity.

As the parcel moves from left to right, it must conserve its mass. As the distance between potential temperature lines increases, the air parcel expands vertically. The area of the air parcel must decrease as the parcel expands vertically if mass is to be conserved. So the value of δA is a function of how quickly the lines of potential temperature change with pressure:

$$\delta A = const * g\left(-\frac{\partial\theta}{\partial p}\right) \qquad (7)$$

Substituting (7) into (6) yields:

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = Const \qquad (8)$$

Equation (8) is the expression for Ertel's Potential Vorticity in isentropic coordinates. The units of (8) are $10^{-6}\,Kkg^{-1}m^2s^{-1}$, which are called PVU (Potential Vorticity Units). The minus sign is installed so that PV is generally positive in the Northern Hemisphere. A value of 2 PVU typically signifies the tropopause (boundary between the troposphere and the stratosphere).

Interpretation of Ertel PV

Eq. (8) essentially describes a ratio of the absolute vorticity of the air parcel $(\zeta_\theta + f)$ to the depth of the vortex $\left(-g\dfrac{\partial\theta}{\partial p}\right)$. *PV must be conserved following the motion in adiabatic, frictionless flow*. So, if the if either the depth or the spin of the air parcel changes, a compensating effect must occur in the other term.

We can further simplify PV by assuming that the atmosphere is incompressible (constant density). If this is the case, the horizontal area δA must be proportional to the depth of the column:

$$\delta A = M(\rho h)^{-1} = Const / h$$

where M = mass of air parcel and h = depth of the air parcel. Both M and ρ are constant in this situation. Subsituting for δA into Equation (6) yields:

$$\frac{(\zeta + f)}{h} = Const \qquad (9)$$

A simple example of flow over a topographic barrier illustrates the impacts that the conservation of PV has on the absolute vorticity of air parcels.

<u>*Westerly Flow Example*</u>

Let's look at an example of this. Suppose we have an air parcel traveling west to east over a mountain, between two potential temperature levels:

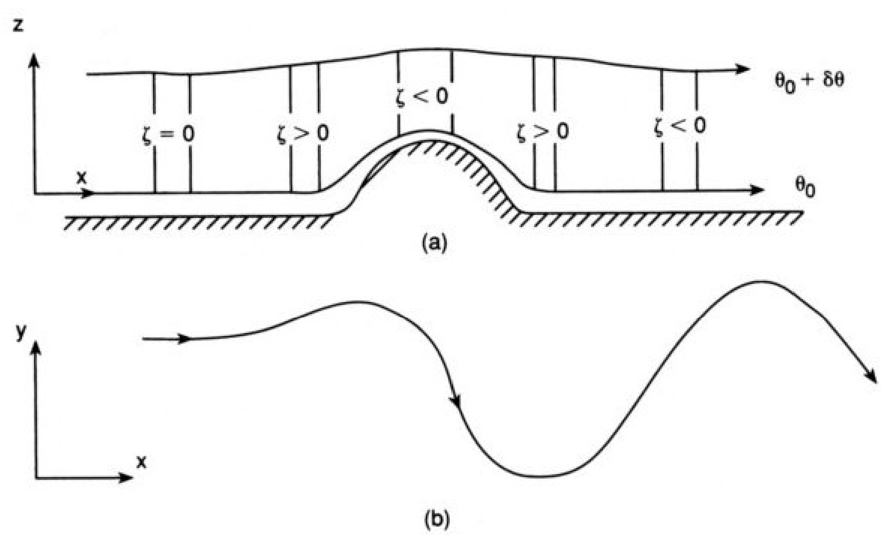
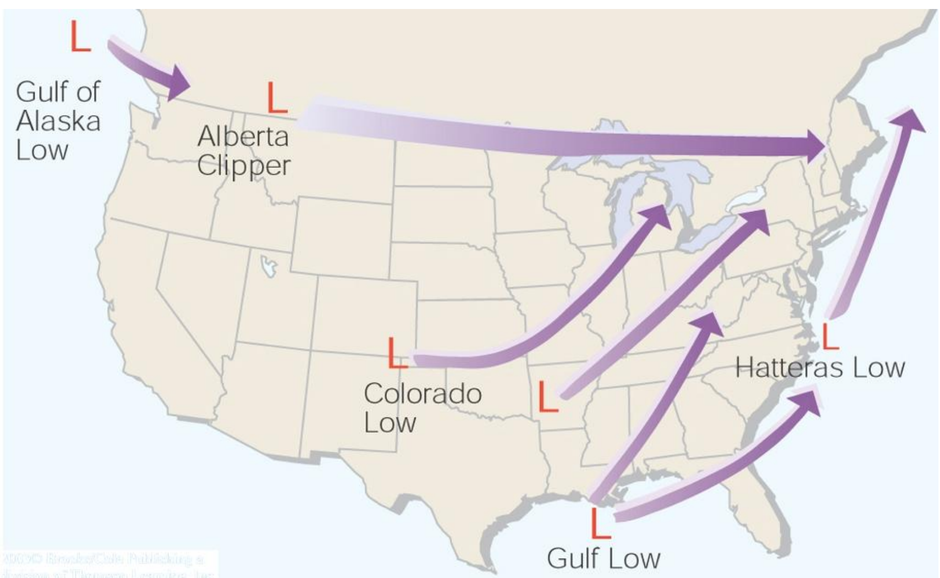


Fig. 4.9 Schematic view of westerly flow over a topographic barrier: (a) the depth of a fluid column as a function of x and (b) the trajectory of a parcel in the $(x,\ y)$ plane.

Near the surface, potential temperature lines follow the surface very closely (see θ_o above). However at upper levels, potential temperature surfaces change altitude farther upstream and downstream of the mountain. So as an air parcel approaches from the left, it initially is stretched vertically, thus h increases and by Eq. (9) $\zeta + f$ must increase to conserve PV. This will cause the air parcel to turn cyclonically as it approaches the mountain, as shown in Fig. 4.9b above (note that as latitude increases, f will increase, thus decreasing the magnitude of ζ necessary to conserve PV).

As the air parcel crosses the mountain, the depth (h) decreases and the vorticity $\zeta + f$ becomes negative. This causes anti-cyclonic turning and southward displacement. When the air parcel returns to its original depth, it will be south of its original latitude. Thus, f will be less and ζ must be positive to conserve PV.

The end result is a wave-like pattern downstream of the barrier, with a lee-side trough followed by an alternating ridge and trough pattern. This is frequently used to explain the tendency of lee-side cyclogenesis.



The figure above shows preferred development locations of mid-latitude cyclones. The lee-side cyclogenesis case would be the “Colorado Low”, located downstream of the Rockies.

Easterly Flow Example

Easterly flow creates a much different downstream flow. Examine the figure below (Fig. 4.10 from Holton):

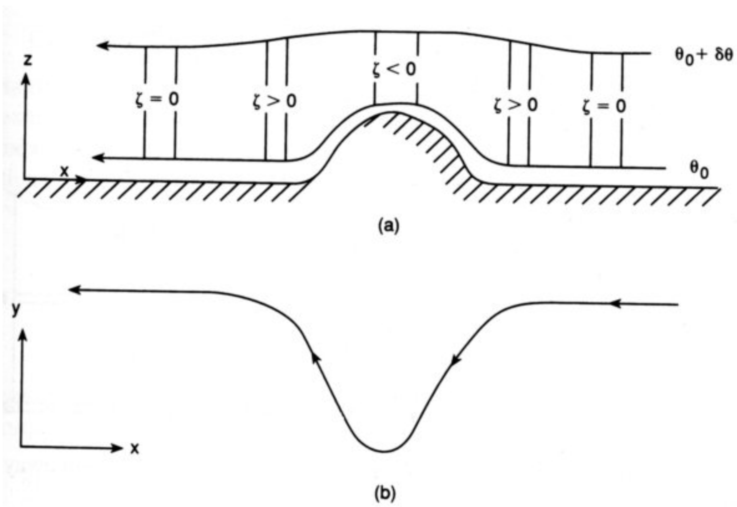


Fig. 4.10 As in Fig. 4.9, but for easterly flow.

A cyclonic curvature is experienced as the air parcel initially stretches, but for easterly flow this means a component of motion *towards* the Equator where f is decreasing. Also, as the parcel approaches the summit the relative vorticity is also decreasing in order to conserve PV since the column is shrinking. Thus, $\zeta + f$ is decreasing both from the change in depth and change in latitude. This produces strong anti-cyclonic curvature as the air parcel passes over the summit. As the parcel again stretches on the lee-side, both ζ and f are becoming more positive and the flow turns more cyclonic. At some distance downstream, the flow will become zonal again – there is no alternating ridge-trough system as was seen in the westerly flow example.

Notes on mountain flows

• In the real atmosphere, vertical motions are generally suppressed and most flows do not go over the top of mountain barriers but around them.

• These examples illustrate the Rossby PV Conservation Law, which states that a change in depth is dynamically analogous to a change in the Coriolis parameter for a barotropic fluid.

Barotropic (Rossby) Potential Vorticity Equation

For an incompressible, barotropic fluid, the absolute vorticity is conserved:

$$\frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0 \qquad (10)$$

The quantity conserved is called the Rossby Potential Vorticity.