Isobaric Coordinates

Before we talk about different forms of balanced flow, we introduce two new coordinate systems: pressure as the vertical coordinate (isobaric coordinate) and the natural coordinate system.

Isobaric Coordinates

So far we have derived relationships using height as a vertical coordinate. Sometimes it is more convenient to use pressure as a vertical coordinate instead of height. Let’s see how by looking at the equations of motion, continuity equation, and thermodynamic equation in isobaric coordinates.

Equations of Motion

Recall the horizontal equations of motion:

\[
\begin{align*}
\frac{du}{dt} &= -\alpha \frac{\partial p}{\partial x} + 2\nu \Omega \sin \phi + F_{rx} \\
\frac{dv}{dt} &= -\alpha \frac{\partial p}{\partial y} - 2\mu \Omega \sin \phi + F_{ry}
\end{align*}
\]  

(1)  

(2)

If pressure is the vertical coordinate, the vertical velocity (w in x,y,z,t coordinate system) is represented as \(\omega\):

\[
\omega = \frac{dp}{dt}
\]

In the isobaric system, \(\omega > 0\) for sinking motion, and \(\omega < 0\) for rising motion. Note that \(\omega\) is different than \(w = \frac{dz}{dt}\).

In an isobaric coordinate system, the total derivative is broken down as (Holton p. 58):

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p}
\]

(3)

Substituting (3) into (1) and (2), we get:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} = -\alpha \frac{\partial p}{\partial x} + f v + F_{rx}
\]

(4)
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} = -\alpha \frac{\partial p}{\partial x} - fu + F_{ry} \quad (5)
\]

We can also transfer the PGF terms into pressure coordinates. We can use the following transformation identity:

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{g}{\alpha} \frac{\partial}{\partial p} \left( \frac{\partial z}{\partial x} \right) \quad (6)
\]

Equation (4) becomes:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} = -\frac{\partial p}{\partial x} - \alpha \left( \frac{g}{\alpha} \frac{\partial p}{\partial x} \right) \frac{\partial z}{\partial x} + fv + F_{rx} \quad (7)
\]

On a constant pressure surface, \( \frac{\partial p}{\partial x} = 0 \) (pressure is not changing). Of course, \( \frac{\partial p}{\partial p} = 1 \).

Equation 7 simplifies to:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} &= -g \frac{\partial z}{\partial x} + fv + F_{rx} \\
\frac{\partial u}{\partial t} &= -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \omega \frac{\partial u}{\partial p} - g \frac{\partial z}{\partial x} + fv + F_{rx} \quad (8)
\end{align*}
\]

The terms can be interpreted as:

a) Rate of change of u with time on a constant pressure surface
b) Zonal advection of u on a constant pressure surface
c) Meridional advection of u on a constant pressure surface
d) Vertical transfer of u between constant pressure surfaces due to rising/sinking air
e) Flow of air vertically along the slope of a constant pressure surface. For example, if
f) Coriolis effect
g) Frictional acceleration

Using the same transformation, equation (5) becomes:

\[
\frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \omega \frac{\partial v}{\partial p} - g \frac{\partial z}{\partial y} - fu + F_{ry} \quad (9)
\]

with the same interpretation of terms as (8) but with the meridional wind rather than
zonal wind.
For vertical motion, we can transform the hydrostatic equation to:

\[
\frac{\partial z}{\partial p} = -\frac{\alpha}{g} \quad \text{and} \quad g \frac{\partial z}{\partial p} = -\alpha \quad \frac{\partial \Phi}{\partial p} = -\alpha
\]

since \(\partial \Phi = gdz\). Substituting from the equation of state,

\[
\frac{\partial \Phi}{\partial p} = -\frac{R_j T}{p} \tag{10}
\]

Continuity Equation

The continuity equation in velocity divergence form (Equation 10 from the Continuity Equation notes) in \(x,y,z,t\) coordinates is:

\[
-\frac{1}{\rho} \frac{d\rho}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{11}
\]

After a ridiculous amount of manipulation and simplification, it can be shown that the continuity equation in isobaric coordinates is:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{12}
\]

This gives arise to one of the reasons why we prefer to use the primitive equations with \(p\) as the vertical coordinate rather than height. Notice that the density term has gone away, so there is now no need to measure it (a hard thing to do). All we need is the various wind components to use the continuity equation. Note also that the time derivative has disappeared as well.

Equation (12) is normally used to compute the vertical velocity between two pressure surfaces. We can move the \(\omega\) term to one side:

\[
\frac{\partial \omega}{\partial p} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{13}
\]

where (a) is the rate of change of vertical velocity between two constant pressure surfaces and (b) is the horizontal convergence in the layer of air between the two surfaces.
Thermodynamic Equation

The thermodynamic equation in (x,y,z,t) coordinates (Equation 2 in thermo_energy notes) is:

\[
\frac{dT}{dt} = \frac{\alpha}{C_p} \frac{dp}{dt} + \frac{1}{C_p} \frac{dq}{dt} \tag{14}
\]

The left hand side (LHS) can be broken down via Euler’s relation, and we can also substitute \( \omega = \frac{dp}{dt} \):

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial z} = \frac{\alpha}{C_p} \omega + \frac{1}{C_p} \frac{dq}{dt} \tag{15}
\]

Move terms (a),(b), and (c) to the right hand side and substitute for \( \alpha \) from the Equation of State:

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - \omega \frac{\partial T}{\partial z} + \frac{R_g T}{p C_p} \omega + \frac{1}{C_p} \frac{dq}{dt} \tag{15}
\]

Eq. 15 is the standard form of the thermodynamic equation in isobaric coordinates.

Interpretation of terms:

- a) Rate of change of \( T \) w/time at a point on a constant pressure surface
- b) Zonal advection of \( T \) on a constant pressure surface
- c) Meridional advection of \( T \) on a constant pressure surface
- d) Vertical transport of \( T \) on a constant pressure surface
- e) Adiabatic \( T \) changes due to expansion/contraction
- f) Diabatic temperature changes (include radiative effects, release of latent heat, etc)

Terms b) through f) all control the temperature change, which in turn controls the thickness of the atmosphere as shown by the hypsometric equation.

On page 59 Holton, he combines terms (d) and (e) of Equation 15:

\[
\frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - \left[ \frac{R_g T}{p C_p} \frac{\partial T}{\partial z} \right] \omega + \frac{1}{C_p} \frac{dq}{dt} \tag{16}
\]

and then defines the static stability parameter (S_p) as:
\[ S_p = \left[ \frac{R_d T}{pC_p} - \frac{\partial T}{\partial z} \right] \omega \]  \hspace{1cm} (17)

Substituting (17) into (16) yields:

\[ \frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - S_p \omega + \frac{1}{C_p} \frac{dq}{dt} \]  \hspace{1cm} (18)

The static stability parameter indicates the stability of the air at a certain pressure level.

If \( S_p < 0 \), then the lapse rate (\( \Gamma \)) is greater than the dry adiabatic lapse rate (\( \Gamma_d \)) and the atmosphere is unstable.
If \( S_p > 0 \), \( \Gamma < \Gamma_d \) (stable)
If \( S_p = 0 \), \( \Gamma = \Gamma_d \) (neutral)