Quasi-Geostrophic (QG) – Situation in which the horizontal velocities are approximately geostrophic.

The QG assumptions make the analysis of extratropical, synoptic-scale motions:

- Simpler to analyze than tropical or planetary scale disturbances
- Able to yield a good approximation of the structure and evolution of ET systems based on the distribution of geopotential height on an isobaric surface

When is it valid to use the QG approximations?

1) If the horizontal acceleration term (e.g. \( \frac{du}{dt} \)) is much less than the Coriolis accelerations:

\[
\frac{du}{dt} \ll f v
\]

If the horizontal accelerations are large, than the wind is not geostrophic and it would not be valid to apply QG theory.

2) Very small Rossby number (\( R_o << 1 \)). Indicates small horizontal accelerations.

3) In general, whenever the flow is near geostrophic balance, we can apply the QG approximation.

The QG Equations

The idea of developing QG equations is to show that the 3-D flow field can be determined approximately by the isobaric distribution of geopotential \([\Phi(x,y,p,t)]\), assuming that the motions are hydrostatic and nearly geostrophic. So we will be working in isobaric coordinates.

First we review the isobaric equations found in Holton section 3.1 and the “Isobaric Coords” section of the .pdf notes:

Horizontal Equations of Motion (Eq. 3.2 p. 58)

\[
\frac{D\vec{V}}{Dt} + \overrightarrow{f}\times \vec{V} = -\nabla_p \Phi \tag{1}
\]

(a) \hspace{1cm} (b) \hspace{1cm} (c)
where a) Total 2-D change of wind, b) horizontal Coriolis force, and c) pressure gradient force.

**Vertical Equation of Motion**

\[
\frac{\partial \Phi}{\partial p} = -\frac{R_d T}{p}
\]  

(2)

**Continuity Equation**

\[
\frac{\partial \omega}{\partial p} = -\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]

(3)

**Thermodynamic Energy Equation**

\[
\frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - S_p \omega + \frac{1}{C_p} J
\]

(4)

Here, \( J = \) diabatic heating \( = \frac{dq}{dt} \)

Recall that the equations listed above had already been simplified to a certain degree. For example, we used the hydrostatic approximation to eliminate vertical velocity terms. We wish to simplify these equations even more by assuming that the flow is nearly geostrophic and that the ratio of the vertical velocity to the horizontal velocity is on the order of \( 10^{-3} \).

**QG Geostrophic Wind**

The total horizontal wind can be separated into “geostrophic” and “ageostrophic” (not geostrophic) components:

\[
\vec{V} = \vec{V}_g + \vec{V}_a
\]

(5)

The geostrophic wind \((V_g)\) is defined as:

\[
\vec{V}_g \equiv f_o^{-1} \vec{k} \times \nabla \Phi
\]

(6)

Here, \( f_o \) is a constant Coriolis parameter (see Holton p. 148 for explanation). Thus the geostrophic wind only depends upon the gradient of geopotential. Of course, \( V_a \) can be found by subtracting the geostrophic wind from the total wind.

In order to validly apply Eq. 6 \( \left| \vec{V}_g \right| >> \left| \vec{V}_a \right| \)
QG Horizontal Equation of Motion

If the ageostrophic component of the wind is not important, then \( \vec{V} \approx \vec{V}_g \). Similarly,

\[
\frac{D\vec{V}}{Dt} \approx \frac{D \vec{V}_g}{Dt} \quad (7a)
\]

where:

\[
\frac{D \vec{V}_g}{Dt} \equiv \frac{\partial}{\partial t} \vec{V}_g + \vec{V}_g \cdot \nabla = \frac{\partial}{\partial t} u_g + \frac{\partial}{\partial x} v_g + \frac{\partial}{\partial y} v_g \quad (7b)
\]

Now, the effect of the variation in the Coriolis parameter must be retained in the derivation of the momentum equation (recall it was assumed constant in the geostrophic wind). The value of \( f \) can be approximated by (through a Taylor series expansion about a reference latitude \( \phi_o \):

\[
f = f_o + \beta y \quad (8)
\]

where \( f_o \) is the Coriolis parameter at the reference latitude and:

\[
\beta \equiv \left( \frac{df}{dy} \right)_{\phi_o} = \frac{2 \Omega \cos \phi_o}{a}
\]

where \( a \) = radius of the earth. This is commonly referred to as the Beta plane approximation.

From Equation (1):

\[
\frac{D\vec{V}}{Dt} + \vec{f} \times \vec{V} = -\nabla_p \Phi \quad (9)
\]

the acceleration following the motion is equal to the difference between the Coriolis force (\( \vec{f} \times \vec{V} \)) and the pressure gradient force (-\( \nabla_p \Phi \)). If these two forces are equal, then

\[
\frac{D\vec{V}}{Dt} = 0 \text{ and there are no horizontal accelerations and the flow is purely geostrophic.}
\]

Therefore, accelerations in the wind must be due to the ageostrophic component of the wind, and we cannot simply replace \( \vec{V} \) with \( \vec{V}_g \) in the Coriolis term in (9). Instead, we substitute Eqs. (5), (7a), and (8) into (9) to yield (see Holton for complete derivation):

\[
\frac{D \vec{V}_g}{Dt} = -f_o \vec{k} \times \vec{V}_a - \beta y \vec{k} \times \vec{V}_g \quad (10)
\]
**QG Continuity Equation**

The geostrophic wind is nondivergent, since it is straight line flow (no directional divergence) and is not accelerating (no speed divergence). So any divergence in the flow must be due to the ageostrophic component of the wind:

\[
\nabla \cdot \vec{V} = \nabla \cdot \vec{V}_a = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \tag{11}
\]

Therefore the continuity equation (3) can be written as:

\[
\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{12}
\]

So the vertical velocity (\(\omega\)) only depends upon the ageostrophic component of the wind.

**QG Thermodynamic Energy Equation**

Starting with the isobaric form of the thermodynamic energy equation:

\[
\left(\frac{\partial}{\partial T} + V_g \cdot \nabla\right)T - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{C_p} \tag{13}
\]

We can substitute for the static stability \((S_p)\) to yield:

\[
\left(\frac{\partial}{\partial T} + V_g \cdot \nabla\right)T - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{C_p}
\]

where \(\sigma \equiv \frac{-RT_o}{p} \frac{d \ln \theta_o}{dp}\). Using the vertical equation of motion for isobaric coordinates (Eq. 2), we can express this equation in terms of the geopotential field:

\[
\left(\frac{\partial}{\partial T} + V_g \cdot \nabla\right)\left(-\frac{\partial \phi}{\partial p}\right) - \sigma \omega = \frac{\kappa J}{p} \tag{13}
\]

where \(\kappa \equiv \frac{R}{C_p}\).